Trakia Journal of Sciences, Vol. 8, No. 3, pp 6-10, 2010

Available online at:
http://www.uni-sz.bg

ISSN 1313-7050 (print)
ISSN 1313-3551 (online)

Original Contribution

MATHEMATICAL FORMULATION OF INDIRECT BOUNDARY ELEMENT METHOD IN CASE OF A WAKE

M. Mushtaq[*], N. A. Shah, G. Muhammad

Department of Mathematics, U.E.T. Lahore, Pakistan

ABSTRACT
In this paper, the mathematical formulation of indirect boundary element method (IBEM) in the case of a wake of a body is presented. When a streamlined body passes through the fluid or the fluid flows past a body at rest. A wake is formed consisting of fluid in regular motion passed near to the boundary of such body and the vorticity is largely confined to the fluid of a wake. For the sake of simplicity, the boundary of the body is discretized into linear quadrilateral elements so that the two-dimensional boundary elements are also linear. The integrals in this case can be evaluated numerically using one-dimensional Gauss-quadrature rule.

Key words: Mathematical Formulation, Indirect boundary element method, Steady flow, Wake.

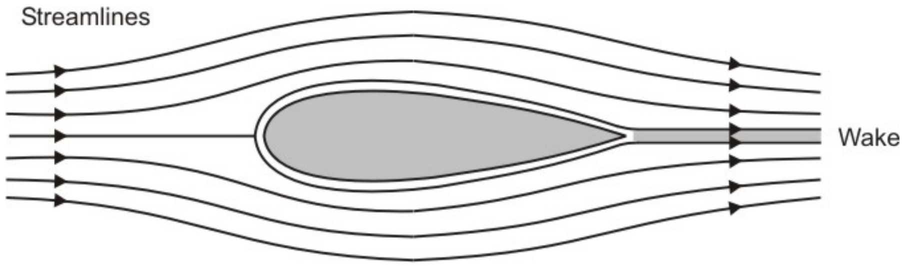
INTRODUCTION

From the time of fluid flow modeling, it had been struggled to find the solution of a complicated system of partial differential equations (PDE) for the fluid flows, which needed more efficient numerical methods. With the passage of time, many numerical techniques such as finite difference method, finite element method, finite volume method and boundary element method etc. came into beings, which made possible the calculation of practical flows. Due to discovery of new algorithms and faster computers, these methods were evolved in all areas in the past. These methods are CPU time and storage hungry. One of the advantages is that with boundary elements one has to discretize the entire surface of the body, whereas with domain methods it is essential to discretize the entire region of the flow field. The most important characteristics of boundary element method are the much smaller system of equations and considerable reduction in data which is prerequisite to run a computer program efficiently. Furthermore, this method is well–suited to problems with an infinite domain. From above discussion, it is concluded that boundary element method is a time saving, accurate and efficient numerical technique as compared to other numerical techniques, which can be classified into direct boundary element method and indirect boundary element method. The indirect method utilizes a distribution of singularities over the boundary of the body and computes this distribution as the solution of integral equation. The derivation of mathematical formulation for such method in the presence of a wake is useful for the solution of physical problems. The term 'wake' is commonly applied to the whole region of non-zero vorticity on the downstream side of the body in an otherwise uniform stream of fluid . The velocity distribution in the wake is likely to be complicated in the neighborhood of the body, even when the flow is steady, judging by the flow fields. The fluid in the region outside of the boundary layer and wake region flows as if it were inviscid. Of course, the fluid viscosity is the same throughout the entire flow field.

***Correspondence to:** *M. Mushtaq, Assistant Professor, Department of Mathematics, U.E.T. Lahore, Pakistan, E-mail: mushtaqmalik2004@yahoo.co.uk*

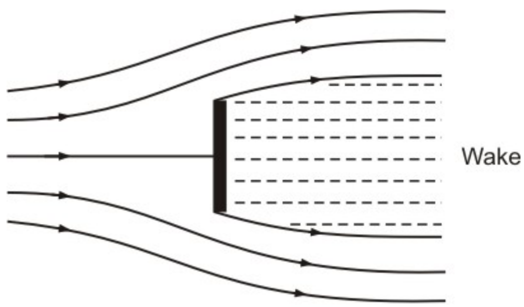
GEOMETRICAL REPRESENTATION OF A WAKE

All the fluids are viscous to some extent and there is a region of the flow field near the body where the viscosity plays a vital role. If the body under consideration is stream lined , then the boundary layer will cover the whole body from the front to the rear end , producing a wake of negligible thickness. While in case of bluff body , the boundary layer separates from the body producing a thick wake.



Flow of a real fluid around a streamlined body

Figure (1)



Flow of a real fluid around a bluff body

Figure (2)

MATHEMATICAL FORMULATION

Let the body under consideration be as shown in **Figure 3** .

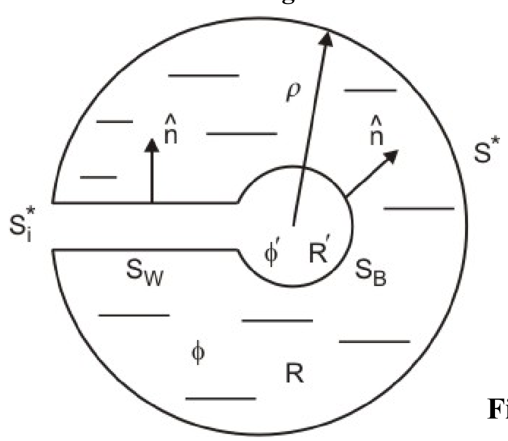


Figure (3)

The surface of body is $S = S_B + S_W + S_i^*$, where S_B is the surface of the actual body, S_W the surface of the tabular wake which starts from the rear end of the body and extends to infinity in the direction of the onset flow and S_i^* the surface of that portion of the sphere S^* cut by the wake of the body.

Let the surface S divide the space into two regions R and R' and let $\hat{n}$ be the outward drawn unit normal to the surface S. Let ϕ and ϕ' denote the velocity potentials of the acyclic irrotational motions in the regions R and R' respectively, with ϕ regular at infinity. Then if the point 'i' be internal to R' and therefore external to R, then we know that the equation of the direct boundary element method for a wake for the region R is

$$c_i\,\phi_i = \phi_\infty - \frac{1}{4\pi}\iint\limits_{S_B} \frac{1}{r}\frac{\partial \phi}{\partial n}\,dS + \frac{1}{4\pi}\iint\limits_{S_B - i} \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS + \frac{1}{4\pi}\iint\limits_{S_W} \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS \tag{1}$$

Now taking $C_i = 0$ then equation (1) becomes

$$0 = \phi_\infty - \frac{1}{4\pi}\iint\limits_{S_B} \frac{1}{r}\frac{\partial \phi}{\partial n}\,dS + \frac{1}{4\pi}\iint\limits_{S_{B-i}} \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS + \frac{1}{4\pi}\iint\limits_{S_W} \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS \tag{2}$$

Further, we also know that the equation of the direct boundary element method for the region R' is

$$c_i\,\phi_i' = \frac{1}{4\pi}\iint\limits_{S} \frac{1}{r}\frac{\partial \phi'}{\partial n}\,dS - \frac{1}{4\pi}\iint\limits_{S-i} \phi'\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS \tag{3}$$

Again taking $c_i = 1$, then equation (3) becomes

$$\phi_i' = \frac{1}{4\pi}\iint\limits_{S} \frac{1}{r}\frac{\partial \phi'}{\partial n}\,dS - \frac{1}{4\pi}\iint\limits_{S-i} \phi'\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS \tag{4}$$

In this region, $S = S_B + S_W + S_i^*$, therefore

$$\begin{aligned}\phi_i' = {} & \frac{1}{4\pi}\iint\limits_{S_B} \frac{1}{r}\frac{\partial \phi'}{\partial n}\,dS + \frac{1}{4\pi}\iint\limits_{S_W} \frac{1}{r}\frac{\partial \phi'}{\partial n}\,dS + \frac{1}{4\pi}\iint\limits_{S_i^*} \frac{1}{r}\frac{\partial \phi'}{\partial n}\,dS \\ & - \frac{1}{4\pi}\iint\limits_{S_B-i} \phi'\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS - \frac{1}{4\pi}\iint\limits_{S_W} \phi'\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS - \frac{1}{4\pi}\iint\limits_{S_i^*} \phi'\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS\end{aligned} \tag{5}$$

Since $\frac{\partial \phi'}{\partial n} = 0$ on S_W, the second integral on the R.H.S. of equation (5) is zero.

Adding equations (2) and (5), then

$$\begin{aligned}\phi_i' = {} & \phi_\infty + \frac{1}{4\pi}\iint\limits_{S_B} \frac{1}{r}\left(\frac{\partial \phi'}{\partial n} - \frac{\partial \phi}{\partial n}\right)dS - \frac{1}{4\pi}\iint\limits_{S_B-i} (\phi' - \phi)\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS \\ & - \frac{1}{4\pi}\iint\limits_{S_W} (\phi' - \phi)\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS + \frac{1}{4\pi}\iint\limits_{S_i^*}\left[\frac{1}{r}\frac{\partial \phi'}{\partial n} - \phi'\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]dS\end{aligned}$$

Similarly in the case when 'i' is internal to R and hence external to R', the same equation results with ϕi, replacing ϕI on the L.H.S. In particular, when 'i' lies on the surface S, then ϕ_i' is replaced by $\frac{1}{2}(\phi_i + \phi_i')$. The above mentioned three cases can be combined by writing

$$\begin{aligned}[c_i\,\phi_i + (1 - c_i)\,\phi_i'] = {} & \phi_\infty + \frac{1}{4\pi}\iint\limits_{S_B} \frac{1}{r}\left(\frac{\partial \phi'}{\partial n} - \frac{\partial \phi}{\partial n}\right)dS \\ & - \frac{1}{4\pi}\iint\limits_{S_{B-i}} (\phi' - \phi)\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS - \frac{1}{4\pi}\iint\limits_{S_W} (\phi' - \phi)\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS \\ & + \frac{1}{4\pi}\iint\limits_{S_i^*}\left[\frac{1}{r}\frac{\partial \phi'}{\partial n} - \phi'\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]dS\end{aligned} \tag{6}$$

where $c_i = \begin{cases} 0 & \text{when 'i' is within R'} \\ 1 & \text{when 'i' is within R'} \\ \frac{1}{2} & \text{when 'i' is on S and S is smooth} \end{cases}$

Let the interior velocity potential ϕ' be equal to the negative of the velocity potential of the uniform stream, $\phi_{u.s}$, then

$$\begin{aligned}c_i\,\Phi_i - (\phi_{u.s})_i = {} & \phi_\infty - \frac{1}{4\pi}\iint\limits_{S_B} \frac{1}{r}\frac{\partial \Phi}{\partial n}\,dS + \frac{1}{4\pi}\iint\limits_{S_{B-i}} \Phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS \\ & + \frac{1}{4\pi}\iint\limits_{S_W} \Phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)dS + \frac{1}{4\pi}\iint\limits_{S_i^*}\left[-\frac{1}{r}\frac{\partial \phi_{u.s}}{\partial n} + \phi_{u.s}\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]dS\end{aligned} \tag{7}$$

Where $\Phi = \phi + \phi_{u.s}$

The boundary condition of zero normal velocity on the body now implies

$\frac{\partial \Phi}{\partial n} = 0$ on S_B, thus the first integral on the R.H.S. becomes zero.

Also the uniform stream in the negative direction of the x–axis has the velocity potential

$$\phi_{u.s} = Ux$$
$$= x \text{ when } U = 1$$

Further, since the outwardly drawn unit normal $\hat{n}$ to S_i^* is in the negative direction of the x–axis, then $\frac{\partial}{\partial n} = -\frac{\partial}{\partial x}$ and the last integral on the R.H.S. of equation (7) becomes

$$\frac{1}{4\pi}\iint_{S_i^*}\left[\frac{1}{r}\frac{\partial x}{\partial x} - x\frac{\partial}{\partial x}\left(\frac{1}{r}\right)\right] dS = \frac{1}{4\pi}\iint_{S_i^*}\left[\frac{1}{r} + x\frac{1}{r2}\frac{\partial r}{\partial x}\right] dS \qquad \Rightarrow 0 \text{ as } -x \text{ and } r \to \infty.$$

Hence equation (7) reduces to

$$-c_i\Phi_i + \frac{1}{4\pi}\iint_{S_B - i}\Phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS + \frac{1}{4\pi}\iint_{S_W}\Phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS + \phi_\infty = -(\phi_{u.s})_i \tag{8}$$

Let the surfaces of the body and the wake be descretized into m quadrilateral elements, then equation (8) can be written in descretized form as

$$-c_i\Phi_i + \frac{1}{4\pi}\sum_{j=1}^{m}\iint_{S_{B_j} - i}\Phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS + \frac{1}{4\pi}\sum_{j=1}^{m}\iint_{S_{B_j}}\Phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS + \phi_\infty = -(\phi_{u.s})_i$$

$$\text{or } -c_i\Phi_i + \sum_{j=1}^{m}\iint_{S_{B_j} - i}\Phi\frac{\partial}{\partial n}\left(\frac{1}{4\pi r}\right) dS + \sum_{j=1}^{m}\iint_{S_{W_j}}\Phi\frac{\partial}{\partial n}\left(\frac{1}{4\pi r}\right) dS + \phi_\infty = -(\phi_{u.s})_i \tag{9}$$

where $S_{B_j} - i$ is the surface area of the element j excluding the point i.

$$\text{Let} \qquad \Phi(\xi, \eta) = \sum_{k=1}^{L} N_k\Phi_k \tag{10}$$

$$\text{Then} \qquad \frac{\partial\Phi}{\partial n}(\xi, \eta) = \sum_{k=1}^{L} N_k\frac{\partial\Phi_k}{\partial n} \tag{11}$$

Where L denotes the number of nodes on each element.

$$\text{Now} \qquad \iint_{S_{B_j} - i}\Phi\frac{\partial}{\partial n}\left(\frac{1}{4\pi r}\right) dS = \iint_{S_{B_j} - i}\left[\sum_{k=1}^{L} N_k\Phi_k\right]\frac{\partial}{\partial n}\left(\frac{1}{4\pi r}\right) dS$$

$$= \iint_{S_{B_j} - i}[N_1\, N_2 \ldots\ldots N_L]\frac{\partial}{\partial n}\left(\frac{1}{4\pi r}\right) dS \begin{Bmatrix}\Phi_1\\ \Phi_2\\ .\\ .\\ .\\ \Phi_L\end{Bmatrix}$$

$$= [h_{ij}^1\; h_{ij}^2 \ldots\ldots h_{ij}^L]\begin{Bmatrix}\Phi_1\\ \Phi_2\\ .\\ .\\ .\\ \Phi_L\end{Bmatrix} \tag{12}$$

$$\text{where} \qquad h_{ij}^K = \iint_{S_{B_j} - i} N_k\frac{\partial}{\partial n}\left(\frac{1}{4\pi r}\right) dS, \quad k = 1, 2, \ldots\ldots, L \tag{13}$$

The h_{ij}^K are influence coefficients defining the interaction between the 'i' and a particular k node on an element 'j'. Similarly,

$$\iint_{S_{W_j}}\Phi\frac{\partial}{\partial n}\left(\frac{1}{4\pi r}\right) dS = \iint_{S_{W_j}}\left[\sum_{k=1}^{L} N_k\Phi_k\right]\frac{\partial}{\partial n}\left(\frac{1}{4\pi r}\right) dS$$

$$= \iint_{S_{W_j}}[N_1\, N_2 \ldots\ldots N_L]\frac{\partial}{\partial n}\left(\frac{1}{4\pi r}\right) dS \begin{Bmatrix}\Phi_1\\ \Phi_2\\ .\\ .\\ .\\ \Phi_L\end{Bmatrix}$$

$$= [W_{ij}^1\; W_{ij}^2 \ldots\ldots W_{ij}^L]\begin{Bmatrix}\Phi_1\\ \Phi_2\\ .\\ .\\ .\\ \Phi_L\end{Bmatrix} \tag{14}$$

where $$W^{k}_{ij} = \iint_{S_{W_j}} N_k \frac{\partial}{\partial n}\left(\frac{1}{4\pi r}\right) dS \quad (15)$$

To write equation (9) corresponding to node 'i', the contribution from all of the elements associated with the node 'i' are to be added into one term, defining the nodal coefficients. This will give the following equation

$$-c_i \Phi_i + \phi_\infty + [\hat{H}_{i1}\, \hat{H}_{i2} \ldots\ldots \hat{H}_{iM}] \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ . \\ . \\ \Phi_M \end{Bmatrix} + [\hat{W}_{i1}\, \hat{W}_{i2} \ldots\ldots \hat{H}_{iM}] \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ . \\ . \\ \Phi_M \end{Bmatrix} = -(\phi_{u.s})_i \quad (16)$$

where each $\hat{H}_{ij}$ and $\hat{W}_{ij}$ term is the sum of the contributions from all the adjoining elements of the node $\hat{i}$. Hence equation (16) represents the assembled equation for node 'i' and can be written as

$$-c_i \Phi_i + \phi_\infty + \sum_{j=1}^{M} \hat{H}_{ij} \Phi_j + \sum_{j=1}^{M} \hat{W}_{ij} \Phi_j = -(\phi_{u.s})_i$$

$$-c_i \Phi_i + \sum_{j=1}^{M} [\hat{H}_{ij} + \hat{W}_{ij}] \Phi_j + \phi_\infty = -(\phi_{u.s})_i$$

or $$-c_i \Phi_i + \sum_{j=1}^{M} H_{ij} \Phi_j + \phi_\infty = -(\phi_{u.s})_i \quad (17)$$

where $$H_{ij} = \begin{cases} \hat{H}_{ij} + \hat{W}_{ij} & \text{when } i \neq j \\ \hat{H}_{ij} + \hat{W}_{ij} - c_i & \text{when } i = j \end{cases} \quad (18)$$

when all the nodes are taken into consideration, equation (17) produces an $M \times (M+1)$ system of equations which can again be put in the matrix form as $[H]\{\underline{U}\} = \{\underline{R}\}$ (19)

where $[H]$ is a matrix of influence coefficients, $\{\underline{U}\}$ is a vector of unknown total potentials Φ_i and $\{\underline{R}\}$ on the R.H.S. is a known vector whose elements are the negative of the values of the velocity potential of the uniform stream at the nodes on the surface of the body.

Note that now $\{\underline{U}\}$ in equation (19) has $(M + 1)$ unknowns $\Phi_1, \Phi_2, \ldots\ldots, \Phi_m, \phi_\infty$. To solve this system of equations, the values of Φ at some position must be specified. For convenience, ϕ_∞ is chosen as zero. Thus the $M \times (M+1)$ system of (equations) reduces to an $M \times M$ system of equations which can be solved easily, but now the diagonal coefficients of $[H]$ will be found by

$$H_{ii} = -\sum_{\substack{j=1 \\ (j \neq i)}}^{M} H_{ij} - 1 \quad (20)$$

The matrix system can again be solved for unknown velocity potentials and thus the pressure coefficients over the surface of the body can be calculated.

CONCLUSION

The mathematical formulation for indirect boundary element method taking into account the wake of the body has been derived. Since the wake is semi-infinite, therefore such formulation can be used to calculate the flow past the semi-infinite bodies. It is also useful to apply this formulation for calculating flow past the bodies of complex shapes such as road vehicles, aeroplanes and ships, etc.

REFERENCES

1. Hess,J.L. and Smith, A.M.O.: "Calculation of non-lifting potential flow about arbitrary three-dimensional bodies", McDonnel Douglas Report No.E.S. 40622 , 1962.
2. Hess, J.L. and Smith, A.M.O.: "Calculation of potential flow about arbitrary bodies", Progress in Aeronautical Sciences, Vol. 8, Pergamon Press, 1967.
3. Hess, J.L.: "Higher order numerical solutions of the integral equation for the two-dimensional Neumann problem", Computer Methods in Applied Mechanics and Engineering, 2: 1973.
4. Morino, L., Chen, Lee-Tzong and Suciu,E.O.: "A steady and oscillatory subsonic and supersonic aerodynamics around complex configuration", AIAA Journal, 13(3): 1975.
5. Milne-Thomson, L.M.: "Theoretical Hydrodynamics", 5th Edition, London Macmillan & Co. Ltd., 1968.
6. Lamb , H. : "Hydrodynamics " 6th Edition , Cambridge University Press,1932 .
7. Batchelor, G.K., F.R.S.: "An Introduction to Fluid Dynamics", Cambridge University Press, 1967.
8. Landahl , M.T. : " Numerical Modeling of blunt-body flows-Problems and Prospects " , Proceeding of the G.M. – Symposim , Aerodynamic Drag Mechanisms of Bluff-Bodies and Road Vehicles, Plenum Press, 1978.
9. Hirt , C.W. and Ramshaw , W.H : "Prospects for numerical simulation of bluff body aerodynamics " , Proceedings of the G.M. – Symposium , Aerodynamics Drag Mechanisms of Bluff – Bodies and Road Vehicles , Plenum Press, 1978.
10. Brebbia, C.A.: "The Boundary element Method for Engineers", Pentech Press, 1978.
11. Brebbia, C.A. and Walker, S.: " Boundary Element Techniques in Engineering" , Newnes-Butterworths, 1980.
12. Shah,N.A., "Ideal fluid dynamics", A-One Publishers, Urdu Bazar, Lahore (Pakistan); 2008.
13. Mushtaq, M.; Shah, N.A. and Muhammad, G.: "Calculation of flow past a sphere in the vicinity of a ground using a direct Boundary Element Method", Australian Journal of Basic and Applied Sciences , 3(2): 480-485, 2009.
14. Mushtaq, M.; Shah, N. A . & Muhammad,G.: "Comparison of Direct and Indirect Boundary Element Methods for the Flow Past a Sphere", Kragujevac Journal of Science,31: 25-32; 2009.
15. Mushtaq, M.; Shah, N.A. and Muhammad, G.: "Indirect Boundary Element Method for Calculation of Pressure Distribution Over the Boundary of a Rankine Half–Body", Global Journal of Pure and Applied Mathematics (in press)